ADVANCED PARTIAL DIFFERENTIAL EQUATIONS: HOMEWORK 2

KELLER VANDEBOGERT

1. Chapter 2, Problem 2

To make this problem a little cleaner to look at, we will employ tensor notation so that $x = (x^1, \ \dots \ , x^n)$ is a vector with components x^i, and the (i, j) component of a matrix will be denoted a^i_j. This also means we use the Einstein summation convention, where the sum is implied if upper and lower indices are the same. With this, let a^i_j be the (i, j) entry of the orthogonal matrix O. We see:

$$v(x^i) = u(y^i)$$

With $y^i := a^i_j x^j$. Then, $\frac{\partial y^i}{\partial x^k} = a^i_j \delta^j_k = a^i_k$. Using this,

$$\frac{\partial v}{\partial x^k} = \frac{\partial u}{\partial y^i} a^i_k$$

And similarly,

$$\frac{\partial^2 v}{\partial x^k \partial x^k} = \frac{\partial^2 u}{\partial y^i \partial y^j} a^i_k a^j_k \tag{1.1}$$

Since O is orthogonal, $O^{-1} = O^T$. In tensor notation, this implies that $\sum_k a^i_k a^j_k = \delta^i_j$, since if the entries of O^{-1} are denoted b^i_j, we see that $a^i_k b^k_j = \delta^i_j$. In the orthogonal case, however, we see that $b^i_j = a^j_i$, hence the above. Now, summing over all k in (1.1), we have:

Date: September 3, 2017.

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\begin{aligned}
\sum_k v_{x^k x^k} &= \Delta v \\
&= \frac{\partial^2 u}{\partial y^i \partial y^j} \sum_k a_k^i a_k^j \\
&= \sum_j \frac{\partial^2 u}{\partial y^i \partial y^j} \delta_j^i \\
&= \sum_j u_{x^j x^j} \\
&= \Delta u = 0
\end{aligned} \tag{1.2}
$$

So that v is harmonic as well, and we are done.

2. Chapter 2, Problem 4

Suppose $U \subset \mathbb{R}^n$. Then, define $u_\epsilon := u + \epsilon|x|^2$. Clearly $u < u_\epsilon$ for all $\epsilon > 0$.

Since u is harmonic, $\Delta u = 0$, implying that $\Delta u_\epsilon = 0 + 2n\epsilon > 0$, and similarly, u_ϵ is continuous and hence attains its maximum and minimum on $\bar{U}$ (since this set is compact).

Now for sake of contradiction, suppose that u attains its maximum on the interior of U. Then, set $M := \max_U u(x)$. Obviously u_ϵ is bounded by $M + \epsilon R$, where $R := \max_{\bar{U}} |x|^2$ (which again exists and is finite as $\bar{U}$ is compact).

Now for $x = (x^1,\ x^2,\ \dots, x^n)$, since $\Delta u_\epsilon > 0$ for all x, we know that $\partial_{x^i x^i} u_\epsilon > 0$ for at least one x^i. However, this then implies that no point of U is a local maximum. Since we know u_ϵ has to obtain a maximum on $\bar{U}$, this then implies that u_ϵ attains its maximum at some $x_0 \in \partial U$.

From here, this implies that $M \leq u_\epsilon(x_0)$, for if $M > u(x_0)$, then $M + \epsilon|y|^2 > M > u(x_0)$ would contradict maximality at x_0, where $u(y) = M$. We then have the following chain of inequalities:

$$u(x) \leq M \leq u_\epsilon(x_0) \leq M + \epsilon R$$

Letting $\epsilon \to 0$, we see that $u_0(x_0) = u(x_0) = M$, contradicting the fact that u attains its maximum at an interior point. Hence we see

$$\max_{\bar{U}} u = \max_{\partial U} u$$

As desired.

3. Chapter 2, Problem 5

(a). Following the proof for the case of harmonic v, define $\phi(r) := ⨍_{B(x,r)} v(y)dy$. Then, we have:

$$\begin{aligned} \phi'(r) &= ⨍_{B(0,1)} v(x+rz) \cdot z dy \\ &= ⨍_{B(x,r)} \frac{\partial v}{\partial \eta} dy \\ &= \frac{r}{n} ⨍_{B(x,r)} \Delta v dy \geq 0 \end{aligned} \tag{3.1}$$

Thus, $\phi'(r) \geq 0$ implies that ϕ is an increasing function. This means that $\lim_{r\to 0} \phi(r) \leq \phi(r_0)$ for all $r_0 \geq 0$. Taking the limit, however:

$$\lim_{r\to 0} \phi(r) = \lim_{r \to 0} ⨍_{B(x,r)} v(y)dy = v(x)$$

Hence $v(x) \leq ⨍_{B(x,r)} v(y)dy$ for any $B(x,r)$, and we are done.

(b). Suppose that v attains its maximum at $x_0 \in U$, $v(x_0) := M$. Then, for $0 < r < \text{dist}(x_0, \partial U)$, use the result of part (a):

$$M = v(x_0) \leq ⨍_{B(x_0,r)} v(y)dy \leq M$$

Which then forces $u(y) = M$ as well for all $y \in B(x_0, r)$. Hence, the set $\{x : V(x) = M\}$ is clopen in the induced topology.

Now, since we've assumed that U is bounded, we know that $\bar{U}$ is compact and hence has finitely many connected components since $\mathbb{R}^n$ is a locally connected space. Hence, v reduces to a constant on a clopen subset of the connected component of x, forcing these two sets to actually be equal.

Thus,

$$\max_{\bar{U}} v = \max_{\partial U} v$$

As desired.

(c). Since ϕ is smooth and convex, $\phi'' \geq 0$. Using this, assume that u is harmonic. Then:

$$\frac{\partial v}{\partial x_i} = \frac{\partial \phi}{\partial u} \frac{\partial u}{\partial x_i}$$
$$\frac{\partial^2 v}{\partial x_i \partial x_i} = \frac{\partial^2 \phi}{\partial u^2} \left(\frac{\partial u}{\partial x_i} \right)^2 + \frac{\partial \phi}{\partial u} \frac{\partial^2 u}{\partial x_i \partial x_i}$$

Summing over all i, we have:

$$\begin{aligned} \Delta v &= \sum_i \frac{\partial^2 \phi}{\partial u^2} \left(\frac{\partial u}{\partial x_i} \right)^2 + \frac{\partial \phi}{\partial u} \Delta u \\ &= \frac{\partial^2 \phi}{\partial u^2} |Du|^2 \geq 0 \end{aligned} \tag{3.2}$$

So that v is subharmonic by definition.

(d). Set $v(x) = |Du|^2 = \sum_j \left(\frac{\partial u}{\partial x_j} \right)^2$. Then, take our first partial:

$$\frac{\partial v}{\partial x_i} = 2 \sum_j \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j}$$

And again:

$$\begin{aligned}\frac{\partial^2 v}{\partial x_i \partial x_i} &= 2\sum_j \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2 + \frac{\partial u}{\partial x_j}\frac{\partial^3 u}{\partial x_i \partial x_i \partial x_j} \\ &= 2\sum_j \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right) + \frac{\partial u}{\partial x_j}\frac{\partial}{\partial x_j}\left(\frac{\partial^2 u}{\partial x_i \partial x_i}\right)\end{aligned} \tag{3.3}$$

Summing over all i, we have:

$$\begin{aligned}\Delta v &= 2\sum_i \sum_j \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2 + \frac{\partial u}{\partial x_j}\frac{\partial^3 u}{\partial x_i \partial x_i \partial x_j} \\ &= 2\sum_i \sum_j \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2 + 2\sum_j \frac{\partial u}{\partial x_j}\frac{\partial}{\partial x_j}\left(\Delta u\right) \\ &= 2\sum_i \sum_j \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2 \geq 0\end{aligned} \tag{3.4}$$

Where the last term is just a sum of nonnegative terms, and is hence nonnegative. Thus by definition, v is subharmonic.

4. Chapter 2, Problem 6

Following the hint, define $\lambda := \max_{\bar{U}} |f|$. Then,

$$\Delta u + \Delta \frac{|x|^2}{2n}\lambda = -f + \lambda \geq 0$$

Hence $v(x) := u + \frac{|x|^2}{2n}\lambda$ is subharmonic and we can employ the result of part (b) of the previous problem to see that v obtains its maximum on the boundary ∂U. Now, since U is bounded, $M := \max_{\bar{U}} |x|^2$ exists and is finite, and it is also clear that $u \leq v$ in U.

Using this,

$$\max_{\bar{U}} |u| \leq \max_{\partial U} |v| \leq \max_{\partial U} |g| + \frac{M}{2n} \max_{\bar{U}} |f|$$

Obviously $\frac{M}{2n} \leq M$, and we can now set $C := \max\{M, 1\}$. This constant is easily seen to satisfy

$$\max_{\bar{U}} |u| \leq C\big(\max_{\partial U} |g| + \max_{\bar{U}} |f|\big)$$

And since M only depends on U, we see that C does as well.